LEIBNIZ AND THE INVENTION OF MATHEMATICAL TRANSCENDENCE. THE ADVENTURES OF AN IMPOSSIBLE INVENTORY

THE DISCOVERY OF THE TRANSCENDENCE

The invention of the mathematical transcendence in the seventeenth century is with good reason, linked to the name of Leibniz. He always claimed this creation – present in his work since 1673. Also one can only truly completely understand Leibniz as a mathematician, through the transcendence, and in relation to this, we can say that in some sense, he embodies it.

Descartes had created a completely new and wide symbolic frame (made of real polynomials with two variables) in which one considers plane curves. Far from reducing the field (as Hofmann wrongly claims it), this allowed the identification for the first time, of a certain concept of *all the curves* (i. e., all algebraic curves). Leibniz found and initially appreciated the Cartesian frame. But, as we shall see in this book, he was presented with, during his research, results and conceptualizations, gradually impossible to express in this context.

Initially, Leibniz qualified them with a vague word, the adjective '*transcendent*', most probably encountered in Nicholas of Cusa. Yet, he never associated this terminology to any philosophical – or theological – connotation; the term simply denoted what surpassed, 'exceeded' the Cartesian frame (it is the etymological meaning of 'transcend') without further constructive definition. Because, for Leibniz, all was first organized within the frame of the Cartesian mathematics, and in relation to it. It happened, however – it was in the nature of things – that what exceeded Descartes would later find some entirely new and diverse mathematical frames, both on a symbolical level (transcendent *expressions* and / or *functions*), a geometrical level (transcendent *curves*) and also a numerical level (transcendent *numbers*). There was only one term, but actually several concepts, which were not necessarily directly connected.

TRANSCENDENCE AND SYMBOLISM

On the symbolic level, the introduction by Newton of exponentials with fractional, then irrational exponents, had performed one of the first breaches in the Cartesian symbolism. In a symbolic approach that would 'transcend' both Descartes and Newton, Leibniz introduced then exponents with letters – unknown or indeterminate – of sign a^x, "exceeding any degree" (as he himself wrote). As we shall see, Leibniz granted too much hope and dedicated too much energy to them. It was what we call here the utopia of *letteralized exponentials* – coextensive with the hope of

Leibniz to have at his disposal an exhaustive inventory of this transcendent field that he brought to light.

A second breach in the Cartesian symbolism had been opened by the expressions of recently discovered functions, such as the logarithm or the exponential, then by obtaining, following Newton, quadratures *via* developments in power series (as $\sqrt{1+x}$). With his – remarkable – arithmetical quadrature of the circle, dated 1673, the young Leibniz, following Mercator, persevered victoriously in this direction by supplying a procedure – through the sum of an infinite series of rational numbers – in order to obtain what is today written as $\pi/4$ (which is a transcendent number).

TRANSCENDENCE AND GEOMETRY

On the geometrical level, the non-Cartesian curves caused a new problem of description for Leibniz; possibly even more complicated than that of symbolism: some well-known curves such as the 'logarithmic' could admittedly have an equation, but this one was not 'Cartesian' (*i. e.* it was not algebraic – in the modern sense). Others, such as the roulette or the spiral (both already rejected by Descartes) might have had a simple geometrical construction, but had no equation (in the Cartesian sense). Some others, such as the catenary or the trochoid, possessed a physical construction (mechanical, often using a thread), but without being provided however, with a Cartesian equation.

On the other hand, however, the introduction of the evolutes of plane curves, invented by Leibniz, following Huygens, allowed him – without significant effort – to mechanically construct transcendent curves from curves that were not transcendent. As it is detailed in this book, Leibniz was interested in examining closely all these eventualities. Leibniz was also the first to introduce the term of *algebraicity* – only with regards to curves – by forming a natural couple of opposites with *transcendence*.

LOOKING FOR AN INVENTORY

In this work, we deal at first with Leibniz's discovery, in the years 1673–1680, of different aspects of the transcendence that we have just mentioned. A decisive turning point took place around 1678. Indeed, in a first step, as we have said, Leibniz had used the term adjectively; he simply wanted to highlight – in diverse situations – what was not Cartesian. In a second step, however, he was not satisfied with the simple observation of various, scattered, established facts, but he adopted in his papers the word 'transcendence' in order to indicate, in its entirety, the territory as new as unknown, of the non-Cartesian entities. If the approach of denomination was positive, it referred to a negatively defined content.

Since the transcendence was the object of a denomination, it was natural that it arose for Leibniz (and incidentally, for John Bernoulli) the issue of its content, that is to say, the extension of the concept – in other words, of an *inventory of the transcendent*. One can understand to what extent this epistemological issue was nurtured by Leibniz's desire of a reasoned inventory of the non-Cartesian field. He wrote for example: "A method of invention would be perfect if we could predict, and even demonstrate, even before entering the subject, what are the ways by which we can reach its completion".

This approach was epistemologically natural, namely to inventory the non-Cartesian field in the same way as Descartes had inventoried the functions and the curves acceptable – according to his views. Retrospectively, it seems to us very naive today. This essay of inventory was indeed complicated by its various instances (symbolical, geometrical, numerical). However, above all, it was in fact made impossible by the completely negative character of the fundamental definition: is transcendent what is not Cartesian. Just like that of the irrational numbers (they are defined as non-rational) the absence of a positive definition of transcendent entities would, in reality, make impracticable any inventory. But, initially, Leibniz was not convinced of such impossibility. And we detail the – numerous and vain – attempts of inventory, in the later studies of Leibniz (1680–1690).

Leibniz's approach towards what he called definitively "the Calculus of Transcendent", however, became an essential part of the Leibnizian calculus. A very important article of Breger, dated 1986, 'Leibniz Einführung of Transzendenten'[1], will be a constant support in this study.

SYMBOLICAL INVENTORY?

In his attempts for a symbolical inventory, Leibniz first wanted to believe that the letteralized exponentials, of sign a^x, exhausted the subject. This is evidenced by numerous early texts. It was a very natural approach, directly following those of Descartes and Newton. Much later, his approach was entirely echoed by John Bernoulli, with his *percurrent calculus*. To a quite natural argument in favour of the exponential, rooted in an analogical symbolism (the 'primacy' of the exponential form), came another argument, significant in Leibniz's eyes: for him, these letteralized exponentials were *soluble* in the differential calculus – a point that we also analyse in detail.

Later, however, Leibniz realized that, so important may have been the creation of exponentials on the epistemological level, these did not fill all the universe of the non-Cartesian symbolism. The issue of knowing whether the percurrent calculus coincided or not with the transcendent calculus came to be therefore, in the forefront, between Leibniz and Bernoulli; in other words, whether or not, the a^x exhausted *all the content* of transcendent expressions. This caused a misunderstanding between both scholars that we will also later detail. More incidentally, Leibniz was also interested in 'intermediary' exponentials, of a particular type, which he called '*interscendent*', of sign $a^{\sqrt{2}}$ for example. These expressions were considered by their creator as 'less transcendent' than the previous ones, and this opened the

1 [Breger 1986].

door to a hierarchy in transcendence, which was pursued by following other approaches.

None of these considerations could however, be decisive: the new transcendent symbolism exceeded all these examples and all these instantiations.

Rather surprisingly, the approach of Leibniz found another type of symbolical completion within the calculus, in an article dated 1694, published in the Journal, des Savants, *Considérations sur la Différence qu'il y a entre l'Analyse Ordinaire et le Nouveau Calcul des Transcendantes* (*Considerations on the Difference Between the Ordinary Analysis and the New Calculus of the Transcendent*), a text of a considerable importance, which we also analyse. The quadratures were the geometrical support, and the sums, the symbolic support. Leibniz's discovery of a common structure to both foundational Triangles, the Arithmetical and the Harmonic, despite the fact that they were structurally opposed, and ultimately because of that very opposition, this led to the construction in the calculus of a harmony by reciprocity, that of the two assemblers of signs d and $\int$. For us, modern mathematicians, this did not (and could not) bring a definitive solution to the issue of the inventory, but, at least, and this time in a permanent way, symbolically constructed a new calculation scheme, based on the differential and integral calculus, and in which the harmony – so dear to Leibniz – was finally restored. In the new Leibnizian calculus, the summation of an arbitrary expression became from now on possible in all cases, the result being a transcendent expression.

GEOMETRICAL INVENTORY?

With regards to geometry, Leibniz began a precise analysis of the Cartesian conception of the 'geometry of the straight line – in order to criticize it. Remember that Descartes had decided to upset the Pappus' classification by completely reconstructing a typology governed by a new conceptualization, through the production of a pair of opposed concepts, specifically Cartesian: geometrical curves versus mechanical curves. In a natural way, Leibniz's criticism focused on the mode of production by Descartes of his geometrical curves. The latter indeed considered, *for every y fixed*, the solutions with respect to x of an equation of type:

$$\mathrm{a}_0\,(y)\,x^{\mathrm{n}} + \mathrm{a}_1(y)\,x^{\mathrm{n}-1}+\ldots+\mathrm{a}_{\mathrm{n}-1}(y)\,x+\mathrm{a}_{\mathrm{n}}\,(y) = 0\ (= \mathrm{F}\,(x, y))$$

where each a_{j} is a real polynomial (F is thus a real polynomial with two variables). This is a very particular conception, insufficiently analysed by commentators, in my opinion. However, as we explained it above, a large number of curves, commonly encountered at that time, escaped from the Cartesian analysis: first the curves that are now said parameterized, to which Leibniz delivered a short and brilliant study in the *Specimen Geometriæ Luciferæ* … But his two most original contributions to the Cartesian criticism were dedicated, on one hand to the evolutes, on the other hand to what he called the 'pointwise' constructions of the transcendent curves.

Regarding the first aspect, Leibniz noticed that the evolute of an algebraic curve is generally a transcendent curve. Therefore, he discovered a remarkable mathematical situation, because it is at the same time similar and reciprocal to that of quadratures: starting from algebraic curves, the new operation – the passage from one curve to its evolute – allows to construct, in a rather systematic way, transcendent curves, as does the squaring – the significant difference is that squaring lies in the field of sums, while the development falls within differentiation; that is to say, the two fundamental reciprocal operations invented by Leibniz.

The second aspect is perhaps less well known. In the early 1690s, Leibniz was interested in a category of transcendent curves that could be ‘constructed’, he said, by means of the ordinary geometry. The paradigm here was the catenary, namely a model of a transcendent curve to which we dedicate below a detailed study in the perspectives of Leibniz. Admittedly, the ‘pointwise’ construction of this one is made by means of the ordinary geometry, provided, however, one adds the additional required knowledge of a numerical quantity, namely the *transcendent number e*. Leibniz was quite aware of this. For him, knowing the value of *e,* was necessary and sufficient to construct the exponential curve, and thereby, in a second step, the catenary. It is thus this quantity, that Leibniz supposes immediately to be transcendent, which is the key according to him, and what he calls the ‘construction of as many points as we want’ of both curves by ordinary geometrical means.

On my part, I wanted to highlight the ontological difference, fundamental in my view, between these two statements: on one hand, ‘points, as much as we want’ and, on the other hand, ‘an arbitrary point’. We already mentioned in this regard that Descartes rejected the quadratrix in the ‘mechanical’ curves, because it was not amenable to the construction of an arbitrary point of it.

I have therefore summarized the contributions of Leibniz in his search for symbolical inventories, as well as geometrical, and for *transcendent expressions,* as well as for *transcendent curves*. In this book, I also examine Leibniz’s contributions to an adequate conceptualization of the *transcendent numbers*. However, one has to ascertain that they were hardly fruitful.

ON HIERARCHIES IN TRANSCENDENCE. EXPLORATIONS BY REPRODUCTION

In the absence of an exhaustive inventory – bringing it to light appeared vainer day after day – geometers such as Leibniz and John Bernoulli dedicated themselves to a systematic and reasoned exploration, however this concerned only a part of the unknown corpus.

We have already noted above an instance concerning interscendent expressions. Another type of approach – more general and epistemologically significant – was to reproduce, by repeating it, a procedure to engender transcendent entities, which were considered more and more complex.

Trying to encompass the complexity of the unknown, through the organization of degrees in this complexity, is indeed a natural epistemological approach. If a procedure A is known to engender a transcendent A(x) from an entity x (which is itself transcendent or algebraic), then, by applying A to A(x), we produce a new transcendent A(A(x)) – it is a new entity that one will be tempted to consider as falling within a *second order of transcendence*; by organizing so an embryonic hierarchical scheme. One can observe the process on the geometrical level in Leibniz, with the successive evolutes: the evolute of an algebraic curve is (generally) transcendent; the evolute of the evolute is then a transcendent curve of second order, etc. We shall widely comment on these practices.

The approach was also dear to John Bernoulli, this time in the symbolic register. He implemented it in 1697, within the frame of his *degrees of percurrence*: since the production of the letteralized exponential a^x generates transcendent, then, for him, the repetition of the same procedure, which gives a^{x^y}, will spontaneously generate transcendent of order two, etc. Later, he attempts to resume the approach, still in the symbolic register; this time, he organizes *degrees of transcendence* through successive quadratures: the quadratrix $A(x)$ of an algebraic function x is (usually) a transcendent function. Under these conditions, the quadratrix $A(A(x))$ produces a transcendent of second order, etc.

RECEPTIONS OF THE TRANSCENDENCE

Introduced by Leibniz, the terminology of the transcendence was resumed by many of his correspondents, and by some influential mathematicians of the time, with the notable exception of Newton. The extreme diversity of protagonists' reactions however, reflects some embarrassment towards the strength and originality of Leibniz's developments; this perplexity was accompanying some hesitations by Leibniz himself regarding the exact, positive extension, of a concept negatively defined. In particular, we fully analyse an instructive controversy between Leibniz and Huygens – the latter even initially refused to accept the concept of transcendence.

Next, we continue the analysis of the reception of transcendence during the eighteenth and nineteenth centuries, both regarding the term and the concept, from Newton to Euler, Lambert, Liouville, and Hilbert. Note that Euler was the first who, in the eighteenth century, fully introduced the Leibnizian vocabulary of transcendence to the mathematical community, through his definitions for functions and curves; he pertinently analysed both aspects. On the other hand, his consideration of transcendent numbers was widely insufficient. Approximately at the same period, Lambert would give their first modern definition: an *algebraic* number is a root of an algebraic equation with integer coefficients; is *transcendent* a number which is not algebraic. We analyse in detail the reasons why Lambert's definition was *symbolically superior* to Euler's, and thus why it is the only one that has survived.

I also underlined that, if the modern concept did exist with Lambert – he was only interested in numbers – he did not always clearly use the term with this meaning. To refer to the modern concept he had uncovered, Lambert first used 'nombre au hasard' (number 'at random') and, concurrently, 'transcendent quantity' – but in fluctuating meanings.

As for numbers, the story of the crossed reception of the term and the concept of transcendence then presented very curious aspects. After Lambert and up till Hilbert, mathematicians would use, instead of 'transcendent number', the periphrasis 'number that is not reducible to algebraic irrationals'. It was only in 1900, in

Hilbert's works, that one could find the full coincidence of the term and the concept – for the numbers. Let us also note that, in the mid-nineteenth century (1844–1851), Liouville gave the first *demonstration* of the existence of transcendent numbers.

In the last part of the book, while studying diverse receptions of Leibniz's work, I dedicated a chapter to Auguste Comte, and to his philosophy of what he calls 'the transcendent analysis'. For Comte, there exists an objective scientific content, namely the transcendent analysis, which nevertheless received in history various successive subjective interpretations, mainly three in number, from Leibniz, Newton, and Lagrange. Next, Comte provides a reasoned history of 'the successive formation of the transcendent analysis', by examining the contributions of these three interpreters. As such, Leibniz is for him the genuine creator of the transcendence; but Comte expresses many reservations regarding the rigor of his process of creation. For him, Newton's conception was certainly more rigorous, however, not free from serious ontological weaknesses. As for Lagrange's approach, in which Comte sees 'the future' of the transcendence, he locates it in the necessary promotion of the 'abstraction' in mathematics – this point is essential for him. He rightly recognizes this aspect in Lagrange's conceptions. But if he insists on this point, he will nevertheless express other reservations on these – they are this time considered too abstract.

In a final chapter, I practiced a – very short – overview of the continuation of the story, namely the modern impacts of Leibniz's definitions of the couple of opposites: 'algebraic / transcendent'. At first, one must highlight this obvious fact: whether about numbers, functions or curves, the distinction is each time performed by characterizing *algebraic objects*. *Transcendent objects* are merely non-algebraic objects. Let me repeat: there is no positive property for characterizing the transcendent functions, namely no definition *for themselves*. Epistemologically speaking, the difficulty is thus obviously the same for demonstrating the irrationality or the transcendence; it is deep-rooted in the lack of definition of the object *per se*.

This rather exceptional situation in mathematics, has led, both for irrationality and transcendence, to use compelled proof methods, which are specific to these two domains, and to which the mathematician is absolutely forced.