I. General notions on magic squares

A *magic square* is a square divided into a square number of cells in which natural numbers, all different, are arranged in such a way that the same sum is found in each horizontal row, each vertical row, and each of the two main diagonals.

A square with n cells on each side, thus n^2 cells altogether, is said to have the *order* n. The constant sum to be found in each row is called the *magic sum* of this square.

Usually what is written in a square of order n, thus with n^2 cells, are the first n^2 natural numbers. Since the sum of all these numbers equals

$$\frac{n^2\,(n^2+1)}{2},$$

the sum in each row, thus the magic sum for such a square, will be

$$M_n = \frac{n(n^2+1)}{2}.$$

I. A square displaying this magic sum in the $2n+2$ aforesaid rows is an *ordinary magic square* (Fig. 1). It meets the minimum number of required conditions, and such a square can be constructed for any given order $n \geq 3$ (a magic square of order 2 is not possible with different numbers).

1	**32**	**34**	**3**	**35**	**6**
30	**8**	**27**	**28**	**11**	**7**
19	**23**	**15**	**16**	**14**	**24**
18	**17**	**21**	**22**	**20**	**13**
12	**26**	**10**	**9**	**29**	**25**
31	**5**	**4**	**33**	**2**	**36**

Fig. 1

But there is an important, other kind.

II. A *bordered magic square* is one where removal of its successive borders leaves each time a magic square (Fig. 2). With an odd-order square, after removing each (odd-order) border in turn, we shall finally reach the smallest possible square, that of order 3. With an even-order square, that will be one of order 4 (its border cannot be removed since there is no magic square of order 2). For order $n \geq 5$, bordered squares are always possible.

16	*81*	*79*	*77*	*75*	**11**	**13**	**15**	**2**
78	**28**	*65*	*63*	*61*	**25**	**27**	**18**	**4**
76	*62*	**36**	*53*	*51*	**35**	**30**	**20**	**6**
74	*60*	*50*	**40**	*45*	**38**	**32**	**22**	**8**
9	**23**	**33**	**39**	**41**	*43*	*49*	*59*	*73*
10	**24**	**34**	*44*	**37**	*42*	*48*	*58*	*72*
12	**26**	*52*	**29**	**31**	*47*	*46*	*56*	*70*
14	*64*	**17**	**19**	**21**	*57*	*55*	*54*	*68*
80	**1**	**3**	**5**	**7**	*71*	*69*	*67*	*66*

Fig. 2

As seen above, the magic sum for a square of order n filled with the n^2 first natural numbers is

$$M_n = \frac{n(n^2+1)}{2}.$$

Clearly, the *average* sum in each case is $\frac{n^2+1}{2}$. Accordingly, for m cells, the average sum should be m times that quantity; this will be called the *sum due* for m cells. Thus, the inner square of order m ($m \geq 5$) within a bordered square of order n must contain in each row its sum due, namely

$$M_n^{(m)} = \frac{m(n^2+1)}{2}$$

if the main square is filled with the n^2 first natural numbers. The sum in a row in one border will therefore differ from the next by n^2+1. From this it follows that pairs of elements of the same border which are horizontally and vertically (diagonally for the corner cells) opposite add up to n^2+1. Such pairs are what we call *complements*: to each 'small' number a (less than $\frac{n^2+1}{2}$) is associated the 'large' number $(n^2+1)-a$. For example, in each border of the above figure the sum of opposite elements is 82, while 41 belongs to the centre and $1, \ldots, 40$ is the set of small numbers. We therefore infer that, for a bordered square of odd order filled with the first natural numbers, the central element must be $\frac{n^2+1}{2}$ (the *median*). The case of even-order squares is similar, except that there are two median numbers, $\frac{n^2}{2}$ and $\frac{n^2}{2}+1$, to be placed in the central 4×4 square, the small numbers being those less than (or equal to) $\frac{n^2}{2}$.

A more particular kind of magic square is that of *composite* squares: the main square comprises subsquares which, taken individually, are also magic (Fig. 3). The possibility of such an arrangement depends on the divisibility of the order of the main square.

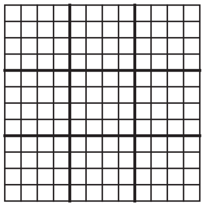

Fig. 3

There exist ways of directly constructing a square of given order and given type, that is, belonging to one of the above two kinds: once the empty square is drawn, a few, easily remembered instructions will enable us to place the sequence of consecutive numbers without any computation or recourse to trial and error. These are called *general construction methods*.

Now there exists no general method *uniformly applicable to any magic square*. Indeed, general methods are applicable to, at most, one of three categories of order, which are:

— The squares of *odd* orders, also called *odd squares*, thus with $n = 2k+1$, of which the smallest is the square of order 3.

— The squares of *evenly-even* orders, also called *evenly-even squares*, thus with $n = 4k$, of which the smallest is the square of order 4.

— The squares of *evenly-odd* orders, also called *evenly-odd squares*, thus with $n = 4k + 2$, of which the smallest is the square of order 6.

These methods are, by definition, applicable whatever the size of the order; but they may require some adapting for squares of lower orders, like 3 or 4, sometimes also 6 and 8.

Remark. Speaking of different *methods* suggests that a square of given order may take different aspects. As a matter of fact, the number of possible configurations (excluding mere inversions and rotations of the square) rapidly increases with the size of the order. Whereas there is just one form of the magic square of order 3, there are already 880 for the order 4, as discovered in 1693.[1]

[1] By Frénicle de Bessy in his *Table generale des quarrez de quatre*.

Note on the history of magic squares

The origin of the science of magic squares is unknown. It is commonly said, and generally accepted, that the earliest magic square appeared in China at the beginning of our era. Now, first, it is the square of order 3, and, second, higher-order squares involving general construction methods do not occur in China before the 12th century, and are clearly of Arabic or Persian origin. For there are from the tenth century onwards numerous texts in Arabic, later also in Persian, displaying a continuous development with earlier methods being extended or new ones invented.

Now trying to fix the starting point of this development led to a major problem: two tenth-century Arabic texts are preserved, one by a competent mathematician, Abū'l-Wafā' Būzjānī (940-997/8) and the other, with only a part on magic squares, by a less competent one, 'Alī ibn Aḥmad al-Anṭākī (d. 987). Their edition, though, shows that their state of knowledge was just in reverse proportion: whereas Būzjānī seemed to write the first steps in the construction of magic squares, and had only one general method, namely for constructing bordered odd-order squares, the second had general methods for all three types of bordered squares and applied them to quite elaborate cases. As the present author wrote in the edition of these texts, *the tenth century, for the* (mediaeval) *history of magic squares, gives the impression of being both a beginning and an end.*[2] The discovery of the text studied here, a translation of an anonymous Greek text, brought the solution: Anṭākī merely reproduced it, and, since it does not, for the case of bordered squares, give the foundation of these general methods, Būzjānī attempted to do so (succeeding only in the case of the odd orders).

As to the question of the first studies on magic squares, it is still unresolved, for those have yet to come to light — and probably never will. Our text does show that already in antiquity elaborate methods were being invented. But it leaves us in ignorance of the earlier history. Besides being anonymous, it does not refer to any person or treatise. We are only told about the existence of ways to construct ordinary magic squares, moreover not considered by our text (see its § 2, or below, p. 165). In short, apart from this isolated treatise and this scarce information, we know nothing about the studies of magic squares in Greek antiquity.

Returning to Arabic times, we do see continuous development in the 11th and early 12th centuries, with the discovery of various constructions for ordinary magic squares. Būzjānī spares no effort in attempting to

[2] *Magic squares in the tenth century*, p. 8.

construct a few ordinary squares of small orders by individual methods, but without finding any general method. Indeed, general methods only begin to appear in the 11th century, first for odd and evenly-even orders; here the contribution of Ibn al-Haytham (ca. 965–1041) has proved to be essential. He considered the properties of the *natural squares*, that is, of the squares filled with the natural numbers taken in succession, and found that their main diagonals give the magic sum for the magic square of the same order while the sums displayed in opposite (horizontal and vertical) rows differ from the magic sum by equal amounts, but with opposite sign. Thus, rather than considering the squares by individual order, he sought (and found) ways to compensate the differences with exchanges between opposite rows. The less simple case of evenly-odd orders was finally solved towards the end of the 11th century.[3]

Squares filled with non-consecutive numbers also began to appear at that time. The origin of that has to do with the association of Arabic letters with numerical values — an adaptation of the Greek numerical system (Fig. 4); this adaptation appeared in early Islamic times, before the adoption of Indian numerals, but remained in use later. Thus to the letters of a word or the words of a sentence can be associated a set of numbers. So, assuming that the word or sentence is written in one row of a square (of which it thus determines the order), the task was then to complete the square numerically so that it would display in each row the sum in question — a mathematically interesting problem since this is not always possible.

ᾱ ا 1	β̄ ب 2	γ̄ ج 3	δ̄ د 4	ε̄ ه 5	ϛ̄ و 6	ζ̄ ز 7	η̄ ح 8	ϑ̄ ط 9
ῑ ى 10	κ̄ ك 20	λ̄ ل 30	μ̄ م 40	ν̄ ن 50	ξ̄ س 60	ō ع 70	π̄ ف 80	ϙ̄ ص 90
ρ̄ ق 100	σ̄ ر 200	τ̄ ش 300	ῡ ت 400	φ̄ ث 500	χ̄ خ 600	ψ̄ ذ 700	ω̄ ض 800	ϡ̄ ظ 900
͵ᾱ غ 1000								

Fig. 4

[3] About this development, in particular how general methods were discovered, and contributions of Ibn al-Haytham and his successors, see our *Magic squares, their history and construction from ancient times to AD 1600* (hereafter simply '*Magic squares*'), pp. 25–29, 51–56, 88–93; or earlier editions: *Les carrés magiques*, pp. 25–28, 49–51, 85–89; *Маг. квадраты*, pp. 33–37, 58–61, 96–100.

Whether original or inspired by earlier texts, quite an elaborate study on the construction of such squares with a set of given numbers not in arithmetical progression appeared already in the early 11th century. We do not know the author's name, but we do at least know his motivation: as he tells us in the introduction, he devoted himself to the study of magic squares in order to find relief from worries and preoccupations.[4] This, apparently efficient, remedy ultimately resulted in a treatise in which, after teaching various methods for the construction of ordinary squares of odd and evenly-even orders and bordered squares of all three types, he constructs squares of orders 3 to 8 with a given sequence of numbers in a row, these numbers corresponding to sacred names or sentences. He makes no allusion to constructing amulets. But any reader might have thought about such a use.

General studies did not stop after the 11th century, although they tended to consist in finding other, simpler or more elegant methods of constructing squares. The treatises also became more numerous. This meant, however, that they less demanded of readers, the main concern being to teach methods without going into the mathematical background. This reached such a point that, as the 13th-century Persian author ʿAbd al-Wahhāb Zanjānī tells us, he had to write, at the request of some friends, a treatise that was shorter than he would have wished, for they merely wanted to have practical rules for constructing magic squares of any order.[5] And such indeed is the characteristic of later treatises: written in response to public demand, they were to teach methods aiming at results, without bothering the reader with questions of foundation or feasibility.

Meanwhile, however, popular use of magic squares as amulets grew steadily. Authors then followed this trend: many late, shorter texts are just not interested in teaching the construction of magic squares — to say nothing of their mathematical foundation. They give the figures of a few magic squares, most commonly squares of the orders 3 to 9 associated with the seven then known planets (including Moon and Sun) of which they embodied the respective, good or evil, qualities — abundantly described and commented in these texts. The reader is taught on what material and when he is to draw each of such squares; for both the nature of the material and the astrologically predetermined time of drawing are presumed to increase the square's efficacy. It merely remains to put this object in the vicinity of the chosen person, beneficiary or victim. This must have been of great use in solving personal or business problems.

[4] See our *Un traité médiéval*, pp. 21 & (Arabic) 208.

[5] See our *Herstellungsverfahren II, II′*.

Of such kind were the first Arabic texts translated into Latin in 14th-century Spain. A characteristic example of theory and application is the 5×5 square, attributed to Mars, which uses its (mostly unfavourable) properties.[6]

The figure of Mars when unfavourable means war and exactions. It is a square figure, five by five, with 65 on each side. If you wish to operate with it, take a copper plate in the day and hour of Mars when Mars is decreasing in size and brightness, or malefic and retrograding, or in any way unfavourable, and engrave the plate with this figure; and you will fumigate it with the excrement of mice or cats. If you place it in an unfinished building, it will never be completed. If you place it in the seat of a prelate, he will suffer daily harm and misfortune. If you place it in the shop of a merchant, it will be wholly destroyed. If you make this plate with the names of two merchants and bury it in the house of one of them, hatred and hostility will come between them. If you happen to fear the king or some powerful person, or enemies, or have to appear before a judge or a court of justice, engrave this figure as said above when Mars is favourable, in direct motion, increasing in size and brightness; fumigate it with one drachma ($= \frac{1}{8}$ ounce) *of carnelian stone. If you put this plate in a piece of red silk and carry it with you, you will win in court and against your enemies in war, for they will flee at the sight of you, fear you and treat you with deference. If you place it upon the leg of a woman,*[7] *she will suffer from a continuous blood flow. If you write it on parchment on the day and the hour of Mars and fumigate it with birthwort and place it in a hive, the bees will all fly away.*

It was thus the arrival of such texts in late mediaeval Europe which first aroused interest in, and later led to the study of, such squares there. This incidentally explains the use of the term ‘magic’ — formerly also ‘planetary’, which we find still employed by Fermat.[8] One of the two 4×4 squares thus transmitted is that used by Dürer in his *Melencolia* in order to date it (Fig. 5; 15 14 occupy the two median cells on the bottom). As to the mediaeval Arabic (perhaps originally Greek) denomination, ‘Harmonious arrangement of numbers’ (*wafq al-aʿdād*), which had a more mathematical connotation, it remained unknown, as well as the various constructions described in Arabic and Persian manuscripts.

[6] This and other examples in our *Magic squares for daily life*. The magic square in question is that of our figure 19, p. 174 below.

[7] Reverting to evil uses.

[8] *Varia opera mathematica*, p. 176; or *Œuvres complètes*, II, p. 194.

16	**3**	**2**	**13**
5	**10**	**11**	**8**
9	**6**	**7**	**12**
4	**15**	**14**	**1**

Fig. 5

The earlier transmission towards the East was more fruitful. India and China received many more examples of squares, sometimes also construction methods. In the Byzantine Empire, a treatise was written at the very beginning of the 14th century by Manuel Moschopoulos. From a set of examples obviously received from some Persian or Arabic manuscript, he reconstructed a few methods for ordinary squares of odd and evenly-even orders. (He obviously does not know anything about the existence of magic squares in ancient Greece.)

European scholars of the fifteenth and sixteenth century did just the same with the squares received, so these initial studies met limited success. A notable exception is Michael Stifel (1487–1567), who gives instructions for constructing bordered squares of all orders;[9] he does not mention any source, nor does he give himself any credit, and his methods are not exactly like the ones we know from Arabic sources.

European research had thus to start afresh. A return to studying their mathematical foundations was initiated by prominent mathematicians like Fermat and Euler, and this helped to dissociate these squares from their unfavourable reputation. Methods of construction were discovered, sometimes rediscovered. New categories were introduced later, such as the 'bimagic' or 'trimagic' square (one remaining magic when its numbers are replaced by their squares or cubes). Pandiagonal squares, with both main and broken diagonals magic, received greater attention than before.[10] A recurrent problem was that of the number of given possibilities for a given order larger than 4 (above, p. 3, *Remark*), closely linked to that of classifiying magic squares by category — which is still the subject of contemporary research.

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[9] See his *Arithmetica integra*, fol. $24^{v}-30^{r}$; or *Magic squares*, pp. 149–150 & 154, 161–163 & 166–167, 170–171 & 173 (*Les carrés magiques*, pp. 125–126 & 130, 137–139 & 141–142, 146 & 148–149; *Маг. квадраты*, pp. 136 & 141, 148–150 & 153–154, 157–158 & 160–161).

[10] One example below (p. 204). Other Arabic examples are given in our *Magic squares*, see *index* there.